MIT OCW GR PSET 5

1 · In a convenient coord system, the space-time of Earth is approximately:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

· Where M is the Earth's mass.

$\Rightarrow g_{\alpha\beta} = \eta_{\alpha\beta} + 2\phi\, \text{diag}(1,1,1,1)$ where $\phi \equiv GM/r$. For cartesian coords. Assume $\phi \ll 1$.

· The space shuttle orbits the Earth in a circular, equatorial orbit of radius R

a · Use the Geodesic Eqn to show that an orbit which begins equatorial remains equatorial (i.e. $dU^\theta/d\tau = 0$):

Given: $g_{\alpha\beta} = \eta_{\alpha\beta} + 2\phi \cdot \text{diag}(1,1,1,1)$, $\phi \equiv \frac{GM}{r}$

$U^r = U^\theta = 0$ $\quad \vec{u} = (U^t, 0, 0, U^\varphi)$

$\theta = \pi/2,\ r = R$

Geodesic Eqn: $\frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\nu\gamma} u^\nu u^\gamma = 0$

· Look at θ component:

$$\frac{du^\theta}{d\tau} + \Gamma^\theta_{\nu\nu} u^\nu u^\nu = 0$$

$$\rightarrow \frac{du^\theta}{d\tau} + \Gamma^\theta_{tt} u^t u^t + \Gamma^\theta_{\varphi\varphi} u^\varphi u^\varphi + 2\Gamma^\theta_{t\varphi} u^t u^\varphi = 0$$

· Now use Γ identities from Carrol :

$$\Gamma^\theta_{tt} = -\frac{1}{2}(g_{tt})^{-1} \partial_\theta g_{tt}$$

$$= \left(-\frac{1}{2}\right)\left(-\left(1-\frac{2GM}{r}\right)\right)^{-1} \partial_\theta \xrightarrow{0} \cancel{\left(-\left(1-\frac{2GM}{r}\right)\right)}$$

$$\rightarrow \Gamma^\theta_{tt} = 0$$

· Now:

$\Gamma^\theta_{\varphi t} = 0$ since this is a diagonal metric

· Finally:

$$\Gamma^\theta_{\varphi\varphi} = -\frac{1}{2}(g_{\varphi\varphi})^{-1} \partial_\theta g_{\varphi\varphi}$$

$$= -\frac{1}{2}\left(\left(1+\frac{2GM}{r}\right)r^2\sin^2\theta\right)^{-1} \partial_\theta\left(\left(1+\frac{2GM}{r}\right)r^2\sin^2\theta\right)$$

$\rightarrow \Gamma^{\theta}_{\varphi\varphi} \propto \frac{\sin\theta\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta}$ at $\theta = \pi/2$

$\rightarrow \Gamma^{\theta}_{\varphi\varphi} \propto \frac{0}{1} = 0$

. So all the relevant Christoffel's vanish +

$\boxed{dU^\theta/d\tau = 0 \quad \text{for all } t}$ Q.E.D. W

b . Find an expression for $\Omega \equiv \frac{d\phi/d\tau}{dt/d\tau}$ and compare to Newton:

Given $\vec{u}\cdot\vec{u} = g_{\alpha\beta}\, U^\alpha U^\beta$, $\vec{u}\cdot\vec{u} = -1$

$\rightarrow -1 = g_{tt}(U^t)^2 + g_{\varphi\varphi}(U^\varphi)^2 + 2\cancel{g_{t\varphi}}^{\,0}\, U^t U^\varphi$

$-1 = -(1-2\phi)\left(\frac{dt}{d\tau}\right)^2 + (1+2\phi)\, r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$

$$\frac{(1-2\phi)(dt/d\tau)^2 - 1}{(1+2\phi)R^2} = (d\phi/d\tau)^2$$

. since $R \gg 1 \Rightarrow$ the -1 term in the LHS numerator does not contribute much, + neather does the $+1$

$$\rightarrow \left(\frac{d\phi}{d\tau}\right)^2 \approx \frac{|-2\phi|\,(dt/d\tau)^2}{(1+2\phi)R^2}$$

• Sub back in $\phi \equiv GM/R$

$$\rightarrow \left(\frac{d\phi}{d\tau}\right)^2 \approx \frac{\frac{2GM}{R}\,(dt/d\tau)^2}{R^2 + 2GMR}$$

• The R^2 term in the denominator dominates

so:

$$\boxed{\frac{d\phi/d\tau}{dt/d\tau} \approx \sqrt{\frac{2GM}{R^3}}}$$ which is similar to

the Newtonian precession $\frac{d\phi}{d\tau} = \sqrt{\frac{GM}{R^3}}$ ✓

$\boxed{c}$ • Using the equation of Geodesic deviation, work out differential equations for the evolution of ξ^x, ξ^y, ξ^z as functions of time for an astronaut who releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x^i_{garbage} - x^i_{shuttle}$. Neglect terms in $\left(\frac{GM}{r}\right)^2$

and treat all orbital velocities as non-relativistic.
Use cartesian coords s.t:

$$x = R\cos(\Omega t),\ y = R\sin(\Omega t),\ \partial_t z = 0$$

- Rather than needing to calculate 256 Riemann components here, the trick is to notice that when we are in the non relativistic limit: (in units where $c \neq 1$)

$$\frac{d\vec{x}}{d\tau} = \vec{u} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$$

$$c\frac{dt}{d\tau} \gg \frac{di}{d\tau} \quad \text{for } i = x, y, z$$

$$\Rightarrow \vec{u} \approx \left(c\frac{dt}{d\tau}, \vec{0}\right) \quad \text{and } \gamma = 1 \text{ so:}$$

$\vec{u} \approx (c, \vec{0})$ so back in units where $c = 1$:

$$\vec{u} \approx (1, \vec{0})$$

- So the full equation of Geodesic deviation:

$$\frac{D^2 \xi^\alpha}{d\tau^2} = R^\alpha{}_{\nu\beta\gamma}\, u^\nu u^\beta \xi^\gamma$$

reduces to $\rightsquigarrow$ $\dfrac{D^2 \xi^\alpha}{d\tau^2} = R^\alpha{}_{00\gamma}\, u^0 u^0 \xi^\gamma$

$$\frac{D^2 \xi^\alpha}{d\tau^2} = R^\alpha_{\ 00\gamma}\, \xi^\gamma$$ and now use the fact

that: $R^\alpha_{\ \nu\beta\gamma} = -R^\alpha_{\ \nu\gamma\beta}$

$$\rightarrow \frac{D^2 \xi^i}{d\tau^2} = -R^i_{\ 0j0}\, \xi^j$$ since we restrict ξ to just spatial separations since we are non-relativistic so our clocks should all be going at the same rate...

· Now we use the definition of the Riemann curvature tensor:

$$R^\alpha_{\ \nu\lambda\sigma} \equiv \partial_\lambda \Gamma^\alpha_{\sigma\nu} - \partial_\sigma \Gamma^\alpha_{\lambda\nu} + \Gamma^\alpha_{\lambda\nu}\Gamma^\nu_{\sigma\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\nu_{\lambda\nu}$$

<u>So;</u>

$$R^i_{\ 0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{j0} + \Gamma^i_{j\nu}\Gamma^\nu_{00} - \Gamma^i_{0\nu}\Gamma^\nu_{j0}$$

· since $\Gamma^i_{jk} = 0$ for diagonal metric with $i \neq j \neq k$

$$\Rightarrow R^i_{\ 0j0} = \partial_j \Gamma^i_{00} - \cancelto{0}{\partial_0 \Gamma^i_{j0}} + \Gamma^i_{jj}\Gamma^j_{00} + \Gamma^i_{ij}\Gamma^i_{00} - \dots$$

$$\ldots \Gamma^i_{00}\Gamma^0_{j0} - \cancelto{0}{\Gamma^i_{i0}\Gamma^i_{j0}}$$

· So now we need to calculate a whole bunch of Christoffel's:

$$\Gamma^i_{00} = \left(-\tfrac{1}{2}\right)(g_{ii})^{-1}\partial_i g_{00}$$

$$g_{ii} = 1 + 2\phi \quad \forall i$$

$$\partial_i g_{00} = \partial_i(-(1-2\phi))$$

· Aside

$$\phi \equiv \frac{GM}{r} = \frac{GM}{\sqrt{x^2+y^2+z^2}}$$

$$\to \partial_i \phi = \left(-\tfrac{1}{2}\right)\frac{2GMi}{(x^2+y^2+z^2)^{3/2}} = \left.\frac{-GMi}{r^3}\right|_{r=R} = \frac{-GMi}{R^3}$$

$$\to \partial_i g_{00} = 2\partial_i\phi = -2GMi/R^3$$

$$\to \Gamma^i_{00} = \left(-\tfrac{1}{\cancel{2}}\right)\left(\frac{1}{1+2\phi}\right)\left(-\cancel{2}GMi/R^3\right) \approx \frac{GMi}{R^3}$$

· Now just Γ^i_{jj} and Γ^i_{ij}:

$$\Gamma^i_{jj} = \left(-\tfrac{1}{2}\right)(g_{ii})^{-1}\partial_i g_{jj} = \left(-\tfrac{1}{2}\right)\left(\frac{1}{1+2\phi}\right)\partial_i(1+2\phi) \approx \frac{GMi}{R^3}$$

as well...

Now:

$$\Gamma^{i}_{ij} = \partial_j \left(\ln \left(\sqrt{|g_{ii}|} \right) \right)$$

$$= \frac{1}{\sqrt{|g_{ii}|}} \cdot \frac{1}{2} \left(|g_{ii}| \right)^{-1/2} \cdot \partial_j g_{ii}$$

$$= \frac{1}{2(1+2\phi)} \cdot \partial_j (1+2\phi) = \frac{\partial_j \phi}{1+2\phi}$$

$$= \frac{-GMj/R^3}{1+2GM/R} \approx \frac{-GMj}{R^3}$$

· Also:

$$\Gamma^{0}_{0j} = \partial_j \left(\ln \left(\sqrt{(1-2\phi)} \right) \right)$$

$$= \frac{1}{\sqrt{1-2\phi}} \cdot \frac{1}{2} \cdot (1-2\phi)^{-1/2} \cdot \left(-2\partial_j \phi \right) \approx -GMj/r^3$$

· So overall:

$$R^{i}_{0j0} \approx \partial_j \left(\frac{GMi}{r^3} \right) + \left(\frac{GM}{r^3} \right)^2 ij - \cancel{\left(\frac{GM}{r^3} \right)^2 ij} + \cancel{\left(\frac{GM}{r^3} \right)^2 ij}$$

· switch to new notation

$$i \to x^i, \quad j \to x^j$$

$$\rightarrow R^{i}_{\ 0j0} \approx \underbrace{\partial_j\left(\frac{GMx^i}{r^3}\right)}_{\substack{\| \\ \underbrace{GM\left(\frac{\delta^{ij}}{r^3} + x^i\partial_j\left(i^2+j^2+k^2\right)^{-3/2}\right)}_{\substack{\| \\ \underbrace{GM\left(\frac{\delta^{ij}}{R^3} + x^i\left(-\frac{3}{2}\right)2x^j\left(i^2+j^2+k^2\right)^{-5/2}\right)}_{\substack{\| \\ GM\left(\frac{\delta^{ij}}{R^3} - \frac{3x^ix^j}{R^5}\right)}}}}}} + \left(\frac{GM}{r^3}\right)^2 x^ix^j$$

$$\rightarrow R^{i}_{\ 0j0} \approx GM\left(\frac{\delta^{ij}}{R^3} - \frac{3x^ix^j}{R^5} + \cancel{\frac{GM}{R^6}x^ix^j}\right)$$

$\emptyset$ since a lot smaller than other terms

$$\rightarrow R^{i}_{\ 0j0} \approx \frac{GM}{R^3}\left(\delta^i_j - \frac{3x^ix_j}{R^2}\right)$$

$$\rightarrow \boxed{\frac{D^2\xi^i}{d\tau^2} = \frac{GM}{R^3}\left(\frac{3x^ix_j}{R^2} - \delta^i_j\right)\xi^j}$$

d . suppose the I.C. are:

$\xi^x = \xi^y = 0$ and $\xi^z = L$, $d\xi^i / d\tau = 0$. Has the astronaut succeeded in getting rid of the garbage?

$$\rightarrow \frac{D^2 \xi^x}{d\tau^2} = \frac{GM}{R^3}\left(\frac{3x}{R^2}\left(x\cancelto{0}{\xi^x} + y\cancelto{0}{\xi^y} + z\xi^z\right) - \cancelto{0}{\xi^x}\right)$$

$$= \left(\frac{GM}{R^3}\right)\left(\frac{3xz}{R^2}\xi^z\right)$$

$$= \left(\frac{3GML}{R^5}\right)xz$$

. but assuming $(x, y, z) = (R\cos(\Omega t), R\sin(\Omega t), 0)$

$$\rightarrow \boxed{\frac{D^2\xi^x}{d\tau^2} = 0}$$

. Likewise, $\boxed{\frac{D^2\xi^y}{d\tau^2} = 0}$

. However, for ξ^z:

$$\frac{D^2 \xi^z}{d\tau^2} = \left(\frac{GM}{R^3}\right)\left(\frac{3z}{R^2}\left(\cancel{x\xi^x}^{\,0} + \cancel{y\xi^y}^{\,0} + z\xi^z\right) - \xi^z\right)$$

$$= \frac{-GML}{R^3} \longrightarrow \boxed{\frac{D^2\xi^z}{d\tau^2} = \frac{-GML}{R^3} \neq 0}$$

- Or more precisely:

$$\frac{D^2\xi^z}{d\tau^2} = -\frac{GM}{R^3}\xi^z \quad \forall t, \text{ but at } t=0,\ \xi^z = L$$

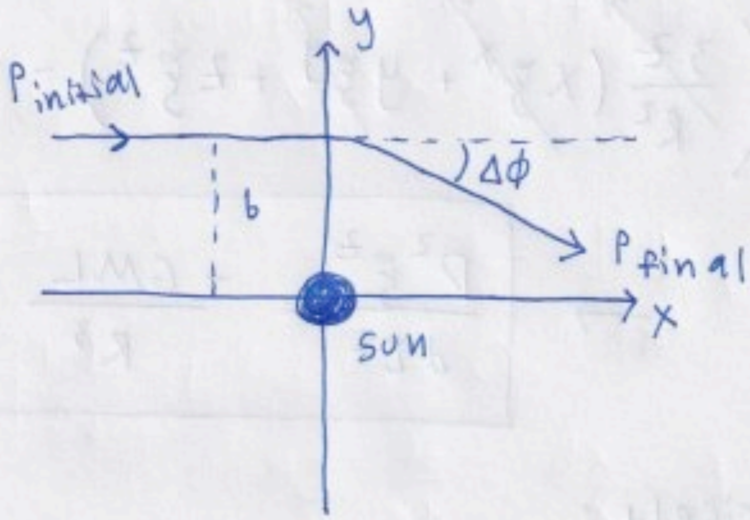
- The solution to such a diff. eq. is: (or at least one type of solution)

$$\boxed{\xi^z = L\cos\left(\sqrt{\frac{GM}{R^3}}\,\tau\right)}$$

- So the astronaut has not succeded since the garbage oscillates up + down around their space station...

[2] · The spacetime of the sun can be written using the same line element from problem 1. Consider a light ray moving parallel to the x-axis with impact parameter from the sun

"b" in the y-direction.



· The problem says to find $\Delta\phi = \left(\frac{p^y}{p^x}\right)_{final}$ but I think it is more straightforward to find $\Delta\phi = \Delta y / \Delta x$. We'll first need to note a few key points to solve this...

· For a photon / lightlike object, the invariant interval ds^2 is:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = c\,dt^2 - dx^2 - \underbrace{dy^2}_{0} - \underbrace{dz^2}_{0}$$

0 approximately at the beginning of the motion...

$$\rightarrow ds^2 = c\,dt^2 - dx^2$$

<u>But</u> $c = dx/dt$

$$\rightarrow ds^2 = c\,dt^2 - c\,dt^2 = 0 \text{ for light !!!}$$

· Also, we can divide by two factors of 'dt' to obtain:

$$0 = g_{\alpha\beta}\frac{dx^\alpha}{dt}\cdot\frac{dx^\beta}{dt} = g_{\alpha\beta}\,u^\alpha u^\beta = \vec{u}\cdot\vec{u}$$

- So for a lightlike photon, $\vec{u}\cdot\vec{u} = \emptyset$ rather than -1! as for a massive particle. Assuming $\vec{u}\cdot\vec{u} = -1$ rather than $\emptyset$ throughout this problem will screw you up!

- Now we assume $u^y = u^z = 0$ (i.e. there is relatively little "speed" in y or z directions compared to the x direction.

- Since we have a photon, $u^x = c = 1$ in natural units:

$$\rightarrow \vec{u} \approx (u^t, 1, 0, 0)$$ ← assuming metric is approximately $\eta_{\alpha\beta}$ "flat"

$$\rightarrow \vec{u}\cdot\vec{u} = 0 = 1 - (u^t)^2 \rightarrow u^t \approx 1$$

$$\rightarrow \vec{u} \approx (1, 1, 0, 0)$$

- <u>Now use</u> the Geodesic equation:

more general form w/o $\lambda = \tau$ ...

$$\frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0 \rightarrow \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\cdot\frac{dx^\nu}{d\lambda} = 0$$

- For massive particles, we normally use $\lambda \rightarrow \tau$ as our affine parameter, but for light-like particles we should use a spatial coord, + since we ultimately want to find dy/dx, we should use $\lambda \to x$:

· <u>so we get</u>:

$$0 = \frac{d^2 y}{dx^2} + \Gamma^{y}_{\nu\nu} \frac{dx^{\nu}}{dx} \cdot \frac{dx^{\nu}}{dx}$$

[redacted]

· Using $\frac{dt}{dx} = \frac{dx}{dx} = 1$, and other

[redacted] terms ≈ 0 we get:

$$0 = \frac{d^2 y}{dx^2} + \Gamma^{y}_{tt} + \Gamma^{y}_{xx}$$

· Now calculate the Γ's:

$$\Gamma^{y}_{tt} = \left(-\tfrac{1}{2}\right)(g_{yy})^{-1} \partial_y g_{tt} \approx \text{[redacted]}\ GMy/r^3$$

$$\Gamma^{y}_{xx} = \left(-\tfrac{1}{2}\right)(g_{yy})^{-1} \partial_y g_{xx} \approx \text{[redacted]}\ GMy/r^3$$

$$\to \left|\frac{d^2 y}{dx^2}\right| \approx \frac{2GMb}{r^3} = \frac{2GMb}{(x^2+b^2)^{3/2}}$$

$$\to \Delta\phi \equiv \frac{dy}{dx} \approx \int \frac{d^2 y}{dx^2} \cancel{dx} \approx 2GMb \int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)^{3/2}}$$

$\rightarrow$ Use $x = b\tan\theta$, $dx = b\sec^2\theta\, d\theta$

$$\Delta\phi \approx 2GMb\int_{-\pi/2}^{\pi/2} \frac{b\sec^2\theta\, d\theta}{b^3(\tan^2\theta+1)^{3/2}}, \text{ use } \tan^2\theta + 1 = \sec^2\theta$$

$$\rightarrow \Delta\phi = \frac{2GMb}{b^2}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{4GM}{b}$$

$$\rightarrow \boxed{\Delta\phi \approx 4GM/b}$$

[3] · Parallel transport on a sphere

· On the surface of a 2-sphere of radius "a", the line element is:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

· consider the vector $A_0 = \vec{e}_\theta$ @ $\theta = \theta_0$, $\phi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ from $0 \le \phi \le 2\pi$. What is the resulting vector $\vec{A}$? What is its magnitude?

· To enforce parallel transport, we assert that the co-variant derivative of A^α w.r.t. φ

is held constant at zero:

$$\nabla_\varphi A^\alpha = \partial_\varphi A^\alpha + \Gamma^\alpha_{\varphi\nu} A^\nu = 0$$

$$\rightarrow \begin{cases} \partial_\varphi A^\varphi + \Gamma^\varphi_{\varphi\nu} A^\nu = 0 \\ \partial_\varphi A^\theta + \Gamma^\theta_{\varphi\nu} A^\nu = 0 \end{cases}$$

$$\rightarrow \begin{cases} \partial_\varphi A^\varphi + \Gamma^\varphi_{\varphi\varphi} A^\varphi + \Gamma^\varphi_{\varphi\theta} A^\theta = 0 \\ \partial_\varphi A^\theta + \Gamma^\theta_{\theta\varphi} A^\theta + \Gamma^\theta_{\varphi\varphi} A^\varphi = 0 \end{cases}$$

· Let's calculate these Gammas:

$$\Gamma^\varphi_{\varphi\theta} = \partial_\theta \ln\sqrt{|g_{\varphi\varphi}|} = \partial_\theta \ln\sqrt{a^2\sin^2\theta}$$

$$= \frac{a\cos\theta}{a\sin\theta} = \cot\theta$$

$$\Gamma^\varphi_{\varphi\varphi} = \partial_\varphi \ln\sqrt{|g_{\varphi\varphi}|} = \partial_\varphi \ln(a\sin\theta) = 0$$

$$\Gamma^\theta_{\theta\varphi} = \partial_\varphi \ln\sqrt{|g_{\theta\theta}|} = \partial_\varphi \ln\sqrt{a^2} = 0$$

$$\Gamma^\theta_{\varphi\varphi} = \left(-\frac{1}{2}\right)(g_{\theta\theta})^{-1}\partial_\theta g_{\varphi\varphi} = \left(\frac{-1}{2a^2}\right)\partial_\theta\left(a^2\sin^2\theta\right)$$

$\rightsquigarrow = -\sin\theta\cos\theta$

· <u>So we get:</u>

$$\begin{cases} \partial_\varphi A^\varphi = (-\cot\theta_0) A^\theta & \text{(i)} \\ \partial_\varphi A^\theta = (\sin\theta_0 \cos\theta_0) A^\varphi & \text{(ii)} \end{cases}$$

· This is a set of coupled diff. eqs. we can solve with standard techniques:

$$\partial_\varphi^2 A^\varphi = (-\cot\theta_0)\partial_\varphi A^\theta = (-\cos^2\theta_0) A^\varphi$$

$$\rightarrow A^\varphi(\phi) = c_1 \sin(\varphi\cos\theta_0) + c_2\cos(\varphi\cos\theta_0)$$

· <u>Plug</u> this back into (ii):

$$\partial_\varphi A^\theta = (\sin\theta_0\cos\theta_0)(c_1\sin(\varphi\cos\theta_0) + c_2\cos(\varphi\cos\theta_0))$$

$$\rightarrow A^\theta = (\sin\theta_0)(c_2\sin(\varphi\cos\theta_0) - c_1\cos(\varphi\cos\theta_0))$$

· <u>Now</u> apply I.C.'s:

[redacted] $A^\varphi(\theta=\theta_0, \varphi=0) = 0 = c_2$

$$\rightarrow \; A^{\varphi} = c_1 \sin(\varphi \cos\theta_0)$$

$$A^{\theta} = -c_1 \sin\theta_0 \cos(\varphi\cos\theta_0)$$

$$A^{\theta}(\theta=\theta_0, \varphi=0) = 1 = -c_1 \sin\theta_0$$

$$\rightarrow \begin{pmatrix} A^{\varphi} \\ A^{\theta} \end{pmatrix} = \begin{pmatrix} \dfrac{-\sin(\varphi\cos\theta_0)}{\sin\theta_0} \\ \cos(\varphi\cos\theta_0) \end{pmatrix}$$

$$\boxed{\begin{aligned} \rightarrow \vec{A}(\theta=\theta_0, \varphi=2\pi) &= \cos(2\pi\cos\theta_0)\,\vec{e}_\theta - \frac{\sin(2\pi\cos\theta_0)\,\vec{e}_\varphi}{\sin\theta_0} \\ &\neq \vec{A}(\theta=\theta_0, \varphi=0) \end{aligned}}$$

· However, this difference in $\vec{A}_i \neq \vec{A}_f$ only amounts to a rotation since the norm is the same:

$$\vec{A}_i \cdot \vec{A}_i = g_{\alpha\beta} A_i^{\alpha} A_i^{\beta} = g_{\theta\theta} A_i^{\theta} A_i^{\theta} + g_{\varphi\varphi} \cancelto{0}{A_i^{\varphi} A_i^{\varphi}}$$

$$= a^2 \cdot 1 \cdot 1$$

$$\rightarrow \|\vec{A}_i\| = a$$

· <u>Now</u>

$$\vec{A}_f \cdot \vec{A}_f = g_{\alpha\beta} A_f^{\alpha} A_f^{\beta} = g_{\theta\theta} A_f^{\theta} A_f^{\theta} + g_{\varphi\varphi} A_f^{\varphi} A_f^{\varphi}$$

$$\Rightarrow \vec{A}_f \cdot \vec{A}_f = a^2 \cos^2(2\pi\cos\theta_0) + \frac{a^2 \sin^2\theta_0}{\sin^2\theta_0} \sin^2(2\pi\cos\theta_0)$$

$$= a^2 \left(\cos^2(\sim) + \sin^2(\sim) \right)$$

$$= a^2$$

$$\Rightarrow \boxed{\| \vec{A}_f \| = a = \| \vec{A}_i \| \quad \checkmark \quad QED}$$

4 a . Compute all the non-zero components of the Riemann tensor R_{ijkl} where $(i, j, k, l) \in (\theta, \phi)$ for the surface of a 2-sphere:

. Use the fact that $R_{ijkl} = g_{i\nu} R^{\nu}_{jkl}$

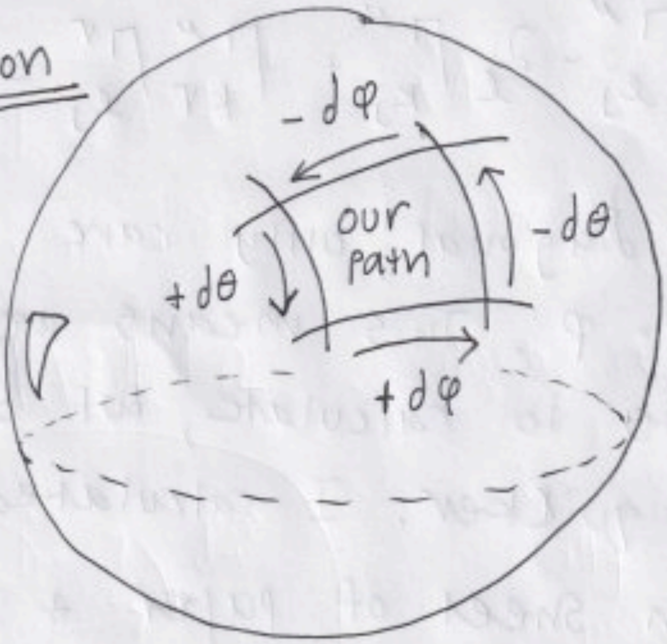
$$\rightarrow R_{ijkl} = g_{i\nu} \left(\partial_k \Gamma^{\nu}_{lj} - \partial_l \Gamma^{\nu}_{kj} + \Gamma^{\nu}_{kr} \Gamma^{r}_{lj} - \Gamma^{\nu}_{lr} \Gamma^{r}_{kj} \right)$$

. Since $[g_{i\nu}]$ is diagonal, only care about $i = \nu = \theta$ and $i = \nu = \varphi$. This means we have 16 (2^4) components to calculate, but symmetry can make this quicker. I calculated them all on a scratch sheet of paper + found the following result: $\rightsquigarrow$

$\boxed{\begin{array}{l} \cdot\ R_{\theta\varphi\theta\varphi} = R_{\varphi\theta\varphi\theta} = a^2 \sin^2\theta \\ \cdot\ R_{\theta\varphi\varphi\theta} = R_{\varphi\theta\theta\varphi} = -a^2\sin^2\theta \\ \cdot\ \text{All other } R_{ijkl} = 0 \end{array}}$

$\boxed{b}$ · Consider the parallel transport of a tangent vector $\vec{A} = A^\theta \vec{e}_\theta + A^\varphi \vec{e}_\varphi$ on this 2-sphere around an infinitessimal parallelogram of sides $\vec{e}_\theta\, d\theta$ and $\vec{e}_\varphi\, d\varphi$. Show that to first order in $d\Omega \equiv \sin\theta\, d\theta\, d\varphi$ the length of $\vec{A}$ is unchanged but its direction rotates through an angle equal to $d\Omega$:

Illustration



· The path we take on our two-sphere

· We know that for a path of a parallelogram on a curved surface, a parallel transported vector's components will change like:

$$\delta A^{\alpha} = R^{\alpha}_{ijk} A^i \delta x^j \delta x^k = g^{\alpha l} R_{lijk} A^i \delta x^j \delta x^k$$

· For a 2-sphere, only $g^{\theta\theta}$ and $g^{\varphi\varphi} \neq 0$ and the 4 R_{ijkl} components we just found

· Using this equation we find that:

"negative part" of our path

$$\delta A^{\theta} = g^{\theta\theta}\left(R_{\theta\varphi\theta\varphi} A^{\varphi} d\theta d\varphi - R_{\theta\varphi\varphi\theta} A^{\varphi} d\theta d\varphi\right)$$

"positive part" of our path

$$\rightarrow \delta A^{\theta} = \left(\frac{1}{a^2}\right)\left(2a^2 \sin^2\theta \, A^{\varphi} d\theta \, d\varphi\right)$$

$$\rightarrow \delta A^{\theta} = \left(2\sin^2\theta\right)\left(d\theta \, d\varphi\right) A^{\varphi}$$

· Also

$$\delta A^{\varphi} = g^{\varphi\varphi}\left(R_{\varphi\theta\varphi\theta} A^{\theta} d\theta d\varphi - R_{\varphi\theta\theta\varphi} A^{\theta} d\theta d\varphi\right)$$

$$= \left(\frac{1}{a^2 \sin^2\theta}\right)\left(2a^2 \sin^2\theta \, d\theta \, d\varphi \, A^{\theta}\right)$$

$$\rightarrow \delta A^{\varphi} = \left(2 \, d\theta \, d\varphi\right) A^{\theta}$$

· Now calculate $\|\delta\vec{A}\|^2$:

$$\|\delta\vec{A}\|^2 = g_{\theta\theta}\,\delta A^\theta \delta A^\theta + g_{\varphi\varphi}\,\delta A^\varphi \delta A^\varphi$$

$$= 4a^2 \sin^4\theta \, d\theta^2 d\varphi^2 A^\varphi A^\varphi + 4a^2 \sin^2\theta \, d\theta^2 d\varphi^2 A^\theta A^\theta$$

· Assuming small θ, $\sin^4\theta \ll \sin^2\theta$

$$\rightarrow \|\delta\vec{A}\|^2 \approx 4a^2 \sin^2\theta \, d\theta^2 d\varphi^2 (A^\theta)^2$$

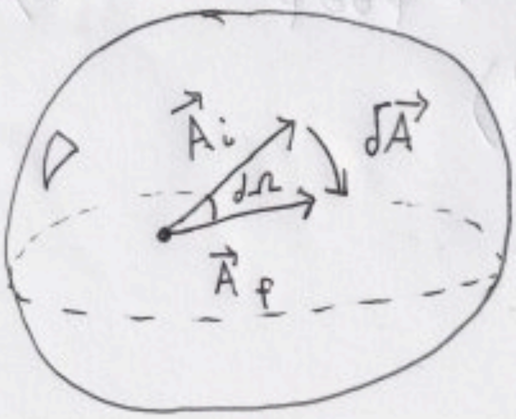
$$\rightarrow \|\delta\vec{A}\| \approx 2a\sin\theta \, d\theta \, d\varphi \, A^\theta = 2aA^\theta \, d\Omega$$

· Now find $\|A^\theta_{initial}\|$ before we parallel transported it:

$$\|A^\theta_i\| = \sqrt{g_{\theta\theta} A^\theta A^\theta} = aA^\theta$$

· So now neglecting the factor of 2:

$\|\delta\vec{A}\| \approx \|A^\theta_i\| \, d\Omega$ which represents a rotation:



· Assuming $d\Omega$ small, this doesn't really affect the overall magnitude (think of an isosceles triangle...)

$\boxed{c}$ · show that if $\vec{A}$ is parallel transported around the boundary of any simply connected solid angle Ω, its direction rotates through an angle Ω:

· I'll do this 2 ways:

1st · We basically derived a diff. eq. in the last part of this problem:

$$\frac{\|\delta \vec{A}\|}{\delta \Omega} \approx \|A_i^\theta\| \longrightarrow$$ when we scale this up for larger paths:

$$\Delta \|\vec{A}\| \approx \|A_i^\theta\| \Omega = \|A_i^\theta\| \Delta\theta$$

· where $\Delta\theta = \Omega$ is the rotation angle.

· A more formal approach might be to look up some formulas + find that for a closed loop γ on a surface enclosing a region R on a curved surface, the net rotation angle of a tangent vector will be:

$$\Delta\theta = \oiint_R K \, dA$$ where $K \equiv$ "curvature of the surface"

and for a sphere $K = 1/r^2$ ...

· since we have $r = a$:

$$\rightarrow \Delta\theta = \frac{1}{a^2} \iint dA = \frac{A}{a^2} \equiv \Omega$$

$\rightarrow$ $\boxed{\Delta\theta = \Omega \text{ is the rotation angle}}$ W

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$\boxed{5}$. Given the following definitions / identities

$$[\nabla_\alpha, \nabla_\beta] V^\nu = R^\nu{}_{\sigma\alpha\beta} V^\sigma$$

$$[\nabla_\alpha, \nabla_\beta] V_\nu = - R^\sigma{}_{\nu\alpha\beta} V_\sigma$$

} Riemann Definitions

$$[\nabla_\alpha, \nabla_\beta] \equiv \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha$$ } Commutator Definition

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$ "Killing's Equation"

- Prove that:

(i) $\nabla_\sigma \nabla_\nu \xi^\alpha = R^\alpha{}_{\nu\sigma\beta} \xi^\beta$

(ii) $\Box \xi^\alpha \equiv \nabla^\nu \nabla_\nu \xi^\alpha = - R^\alpha{}_\beta \xi^\alpha$

- <u>Start</u> with (i)

$$\nabla_\sigma \nabla_\nu \xi^\alpha = [\nabla_\sigma, \nabla_\nu] \xi^\alpha + \nabla_\nu \nabla_\sigma \xi^\alpha$$

$$= \underbrace{R^\alpha{}_{\beta\sigma\nu} \xi^\beta} + \nabla_\nu \nabla_\sigma \xi^\alpha$$ by Riemann Definitions

$$g^{\alpha\omega} R_{\omega\beta\sigma\nu} \xi^\beta = - g^{\alpha\omega} R_{\omega\beta\nu\sigma}$$ by Riemann Symmetries...

$$= - g^{\alpha\omega} R_{\nu\sigma\omega\beta} = + g^{\alpha\omega} R_{\nu\sigma\beta\omega}$$

$$\rightarrow \nabla_\gamma \nabla_\nu \xi^\alpha = R^\alpha{}_{\nu\gamma\beta}\,\xi^\beta + \nabla_\nu \nabla_\gamma \xi^\alpha \qquad (\star)$$

- Now to simplify the last term, look to Killing's equation:

[redacted]

- $\nabla_\gamma \xi_\omega + \nabla_\omega \xi_\gamma = 0$

$$\rightarrow g^{\alpha\omega} \nabla_\gamma \xi_\omega = - g^{\alpha\omega} \nabla_\omega \xi_\gamma$$

$$\rightarrow \nabla_\gamma \xi^\alpha = - g^{\alpha\omega} \nabla_\omega \xi_\gamma$$ Plug back into $(\star)$

$$\rightarrow \nabla_\gamma \nabla_\nu \xi^\alpha = R^\alpha{}_{\nu\gamma\beta}\,\xi^\beta - g^{\alpha\omega} \nabla_\nu \nabla_\omega \xi_\gamma$$

$$= R^\alpha{}_{\nu\gamma\beta}\,\xi^\beta - g^{\alpha\omega}[\nabla_\nu, \nabla_\omega]\xi_\gamma - g^{\alpha\omega}\nabla_\omega \nabla_\nu \xi_\gamma$$

$$= R^\alpha{}_{\nu\gamma\beta}\,\xi^\beta + \underbrace{g^{\alpha\omega} R^\beta{}_{\gamma\nu\omega}\,\xi_\beta}_{\text{Riemann Definition}} - \underbrace{g^{\alpha\gamma} \nabla_\gamma \nabla_\nu \xi_\gamma}_{\text{redefine dummy indices}}$$

$$= R^\alpha{}_{\nu\gamma\beta}\,\xi^\beta + g^{\alpha\omega} R_{\beta\gamma\nu\omega}\,\xi^\beta - \nabla_\gamma \nabla_\nu \xi^\alpha$$

$\hookrightarrow -\gamma\beta\nu\omega$
$\hookrightarrow +\gamma\beta\omega\nu$
$\hookrightarrow +\omega\nu\gamma\beta$

by Riemann Symmetries

$$\rightarrow \nabla_\sigma \nabla_\nu \xi^\alpha = 2R^\alpha{}_{\nu\sigma\beta}\xi^\beta - \nabla_\sigma\nabla_\nu \xi^\alpha$$

$$\rightarrow \boxed{\nabla_\sigma \nabla_\nu \xi^\alpha = R^\alpha{}_{\nu\sigma\beta}\xi^\beta \quad \text{Q.E.D.}}$$

· Now on to prove (ii)

$$\Box \xi^\alpha \equiv \nabla^\nu \nabla_\nu \xi^\alpha$$

$$= g^{\nu\sigma}\nabla_\sigma \nabla_\nu \xi^\alpha$$

$$= g^{\nu\sigma} R^\alpha{}_{\nu\sigma\beta}\xi^\beta \quad \text{by identity (i)}$$

$$= g^{\nu\sigma} g^{\alpha\omega} R_{\omega\nu\sigma\beta}\xi^\beta$$

$$= -g^{\alpha\omega} g^{\nu\sigma} R_{\nu\omega\sigma\beta}\xi^\beta \quad \text{by Riemann Symmetry}$$

$$= -g^{\alpha\omega} R^\sigma{}_{\omega\sigma\beta}\xi^\beta$$

$$= -g^{\alpha\omega} R_{\omega\beta}\xi^\beta$$

$$= -R^\alpha{}_\beta \xi^\beta \longrightarrow \boxed{\Box \xi^\alpha = -R^\alpha{}_\beta \xi^\beta \quad \text{Q.E.D.}}$$

$\boxed{6}$. Compute all the non-vanishing components
$\boxed{a}$ of the Riemann tensor for the spacetime with the line element:

$$ds^2 = -e^{2\phi(x)}dt^2 + e^{-2\psi(x)}dx^2$$

- This is basically just another brute force calculation. Use the fact that:

$$R_{ijkl} = g_{i\nu} R^{\nu}_{\ jkl}$$

$$= g_{i\nu}\left(\partial_k \Gamma^{\nu}_{lj} - \partial_l \Gamma^{\nu}_{kj} + \Gamma^{\nu}_{k\sigma}\Gamma^{\sigma}_{lj} - \Gamma^{\nu}_{l\sigma}\Gamma^{\sigma}_{kj}\right)$$

- And since $[g_{i\nu}]$ is diagonal we only care about $i=\nu=t$ or $i=\nu=x$ where $(i,j,k,l)\in(t,x)$
- This gives us 16 components to calculate, (but less do the symmetries of R_{ijkl}):

$$\begin{Bmatrix} tttt,\ tttx,\ txtt,\ ttxt,\ txxt,\ ttxx,\ txtx,\ txxx; \\ xttt,\ xttx,\ xxtt,\ xtxt,\ xxxt,\ xtxx,\ xxtx,\ xxxx \end{Bmatrix}$$

- I did the calculations on some scratch paper & found the following: $\rightsquigarrow$

$$\boxed{\begin{array}{l} \cdot\ R_{txtx} = R_{xtxt} = e^{2\phi}\left(\partial_x^2\phi + (\partial_x\phi)^2 + (\partial_x\phi)(\partial_x\psi)\right) \\ \cdot\ R_{txxt} = R_{xttx} = -R_{txtx} = -R_{xtxt} \\ \cdot\ \text{All other } R_{ijkl} = 0 \end{array}}$$

[6] · For the case $\phi = \psi = \frac{1}{2}\ln|g(x-x_0)|$ where g and x_0 are constants, show that the space-time is flat + find a coordinate transformation to globally flat coordinates $(\bar{t}, \bar{x})$ s.t. $ds^2 = -d\bar{t}^2 + d\bar{x}^2$

· 1st sub $\phi = \psi = \frac{1}{2}\ln|g(x-x_0)|$ back into our original line element:

$$ds^2 = -e^{\ln|g(x-x_0)|}dt^2 + e^{-\ln|g(x-x_0)|}dx^2$$

$$= -|g(x-x_0)|\,dt^2 + \frac{dx^2}{|g(x-x_0)|}$$

· We want to find $\bar{x}$ such that:

$$(d\bar{x})^2 = \frac{(dx)^2}{|g(x-x_0)|} \rightsquigarrow \frac{d\bar{x}}{dx} = \frac{1}{\sqrt{|g(x-x_0)|}}$$

$$\rightarrow \int d\bar{x} = \int \frac{dx}{\sqrt{|g(x-x_0)|}}$$ Define $u \equiv g(x-x_0)$

$$\rightarrow du = g\,dx$$

$$\rightarrow \bar{x}(x) = \frac{1}{g}\int \frac{du}{\sqrt{|u|}} = \left(\frac{1}{g}\right) 2\sqrt{|u|}$$

$$\rightarrow \boxed{\bar{x}(x) = \frac{2}{g}\sqrt{|g(x-x_0)|}}$$

· <u>Also</u> need $\bar{t}(x,t)$:

· Enforce that: $d\bar{t}^2 = |g(x-x_0)|\,dt^2$

$$\rightarrow \frac{d\bar{t}}{dt} = \sqrt{|g(x-x_0)|}$$

$$\rightarrow \boxed{\bar{t}(x,t) = t\sqrt{|g(x-x_0)|}}$$

· Now with this coordinate transformation,

$ds^2 = -d\bar{t}^2 + d\bar{x}^2$ which is the metric of a $\boxed{\text{flat space-time}}$ ✓✓